Name: ______________________________

Teacher: ______________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2011

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed
Determines the important features of graphs of a wide variety of functions, including conic sections
Applies appropriate algebraic techniques to complex numbers and polynomials
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems
Uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started in a new booklet

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1: **Marks**

a) Find $\int \frac{e^{2x}-1}{e^x-1}dx$ **1**

b) Find $\int \frac{\tan x}{\tan 2x}dx$ **2**

c) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}}dx = \frac{\pi}{6}$ **3**

d) Find $\int \frac{x^2+5x-4}{(x-1)(x^2+1)}dx$ **4**

e) The integral I_n is defined by $I_n = \int_0^1 x^n e^{-x}dx$.

i. Show that $I_n = nI_{n-1} - e^{-1}$. **2**

ii. Hence show that $I_3 = 6 - 16e^{-1}$. **3**

Question 2: **Marks**

a) Given $z=\sqrt{3}-i$:

i. Express z in modulus-argument form. **2**

ii. Hence evaluate $\left(\sqrt{3}-i\right)^6$. **2**

b) $z=1+i$ is a root of the equation $z^2-aiz+b=0$, where a and b are real numbers.

i. Find the values of a and b. **2**

ii. Find the other root of the equation. **2**

c) The complex number z is given in modulus/argument form by $z=r\left(\cos\theta+i\sin\theta\right)$. Show that $\dfrac{z}{z^2+r^2}$ is real. **3**

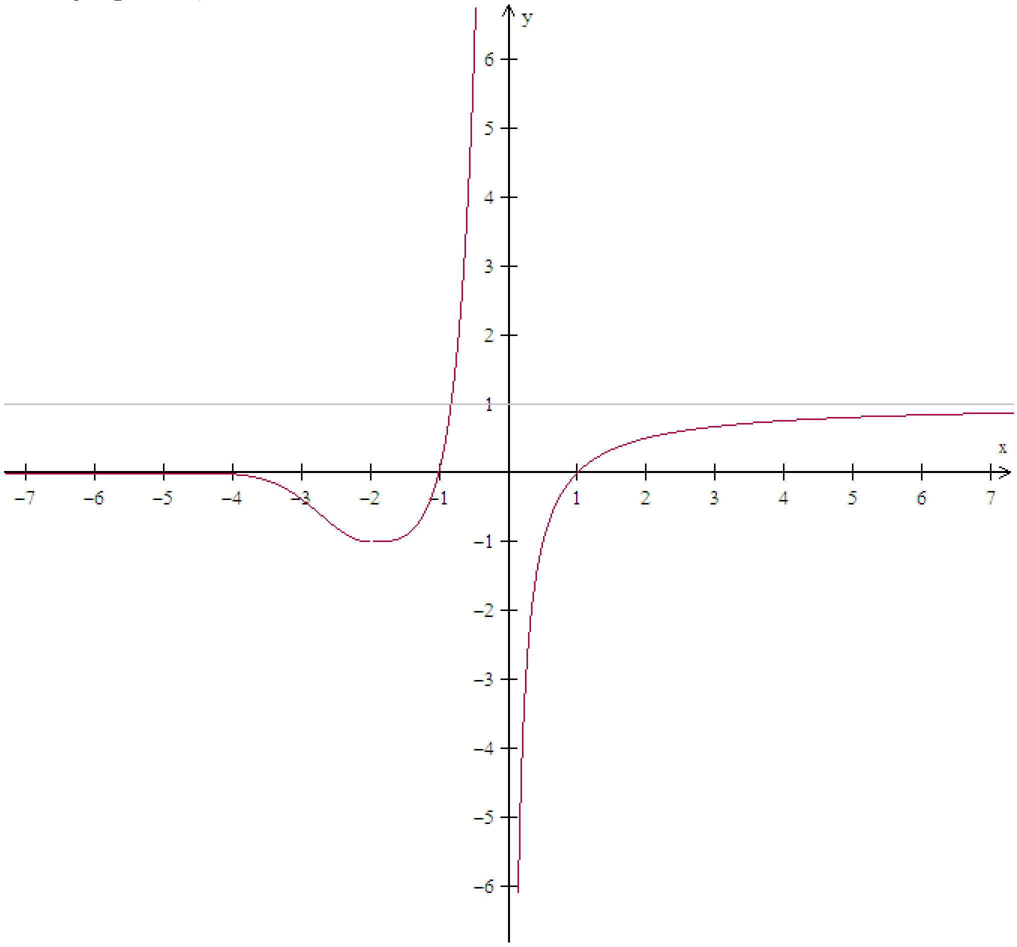
d) The locus of all points z in the complex plane which satisfy $\arg\left(\dfrac{z-3}{z+1}\right)=\dfrac{\pi}{3}$ forms part of a circle.

i. Sketch this locus. **2**

ii. Find the centre and radius of the circle. **2**

Question 3: **Marks**

a) The graph of $y = f(x)$ is shown below. (The lines $y = 1$ and the x-axis are asymptotes.)



Draw a neat one-third page sketch of the following, showing relevant features:

i. $y = |f(x)|$ **1**

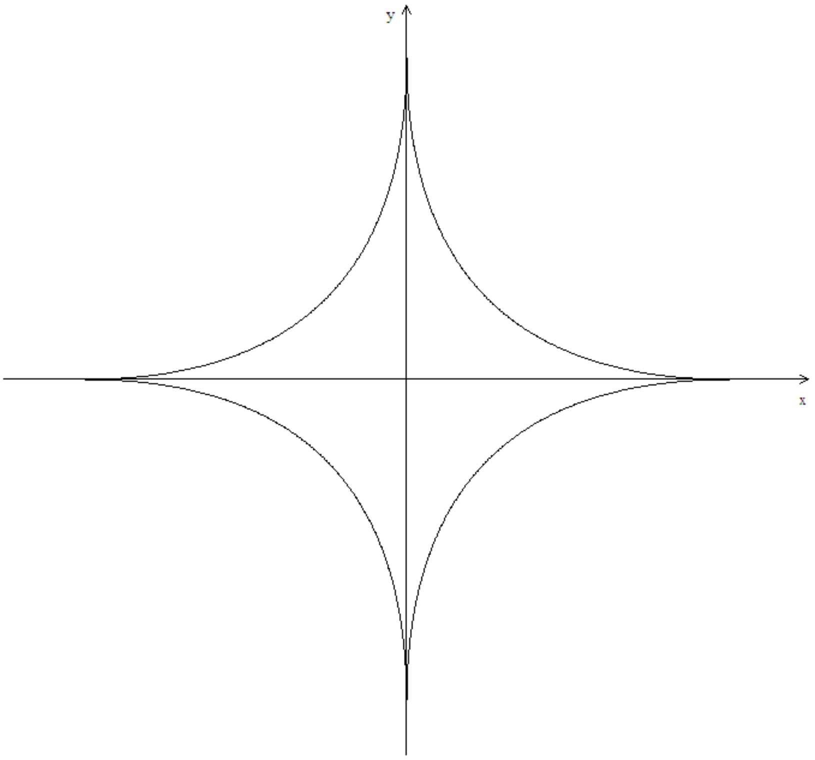
ii. $y = f(-x)$ **1**

iii. $y = \dfrac{1}{f(x)}$ **2**

iv. $y = (f(x))^2$ **2**

v. $y = e^{f(x)}$ **2**

b) The diagram shows the graph of the relation $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = L^{\frac{1}{2}}$, for $L > 0$.



i. Show that the area of the region enclosed by the curve is $\frac{2}{3}L^2$ **3**

ii. A stone column has height H metres. Its base is the region enclosed by the curve $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = L^{\frac{1}{2}}$, and the cross section taken parallel to the base at height h metres is a similar region enclosed by the curve $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = l^{\frac{1}{2}}$ where $l = L\left(1 - \frac{h}{H}\right)$. Find the volume of the stone column (in terms of *L* and *H*). **4**

Question 4: **Marks**

a) It is given that the hyperbola $xy = c^2$ touches (is tangential to) the parabola $y = x - x^2$ at point Q and crosses the parabola again at point R:

i. Show this information on a sketch **1**

ii. Deduce that the equation $x^3 - x^2 + c^2 = 0$ has a repeated root and hence find the value of c^2 **3**

iii. Find the coordinates of point R. **1**

b) $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

i. Show that the equation of the normal to the hyperbola at point P has the equation $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$ **4**

ii. The line through P parallel to the $y-$axis meets the asymptote $y = \dfrac{bx}{a}$ at Q. The tangent at P meets the same asymptote at R. The normal at P meets the $x-$axis at G. Prove that $\angle RQG$ is a right-angle. **2**

c) The region between the curve $y = \sin x$ and the line $y = 1$. From $x = 0$ to $x = \dfrac{\pi}{2}$ is rotated about the line $y = 1$. Using a slicing technique, find the exact volume of the solid thus formed. **4**

Question 5: **Marks**

a)

i. Find the general solution to the equation $\cos 4\theta = \frac{1}{2}$ **2**

ii. Use De Moivre's Theorem to show that
$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ **3**

iii. Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots
$x_1 = \cos\frac{\pi}{12}, x_2 = \cos\frac{5\pi}{12}, x_3 = \cos\frac{7\pi}{12}, x_1 = \cos\frac{11\pi}{12}$ **2**

iv. By considering this equation as a quadratic in x^2, show that
$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$. **3**

b) A particle is moving in the positive direction along a straight line in a medium that exerts a resistance to motion proportional to the cube of the velocity. No other forces act on the particle, i.e. $\ddot{x} = -kv^3$, where k is a positive constant.

At time $t = 0$, the particle is at the origin and has a velocity U. At time $t = T$, the particle is at $x = D$ and has a velocity $v = V$.

i. Using $\ddot{x} = \frac{dv}{dt}$, show that $\frac{1}{V^2} - \frac{1}{U^2} = 2kT$. **2**

ii. Using the identity $\ddot{x} = v\frac{dv}{dx}$, show that $\frac{1}{V} - \frac{1}{U} = kD$. **3**

Question 6: **Marks**

a) The equation $x^3 + kx + 2 = 0$ has roots α, β and γ.

i. Find an expression for $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ in terms of k. **2**

ii. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is independent of k. **2**

iii. Find the monic equation with roots $\alpha^2, \beta^2, \gamma^2$ (leaving coefficients in terms of k). **3**

b) A particle of mass m falls under gravity in a medium whose resistance R to the motion is proportional to the square of the speed $\left(R = mkv^2\right)$. Acceleration due to gravity is g.

i. Find an expression for the terminal velocity V_t in this medium. **2**

A second particle of mass M is projected vertically upward from ground level in the same medium with an initial velocity U. It takes T seconds to reach its maximum height H above the projection point.

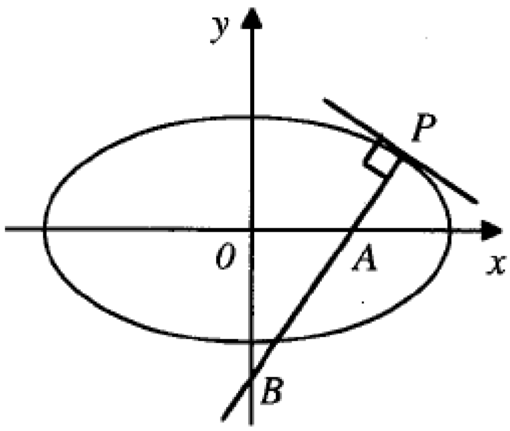
ii. Show that $T = \frac{V_t}{g} \tan^{-1}\left(\frac{U}{V_t}\right)$. **3**

iii. Show that $H = \frac{V_t^2}{2g}\left[\ln\left(\frac{V_t^2 + U^2}{V_t^2}\right)\right]$. **3**

Question 7: **Marks**

a) A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. Find the volume of the solid formed if every section perpendicular to the major axis is an isosceles triangle with altitude 6 units. **3**

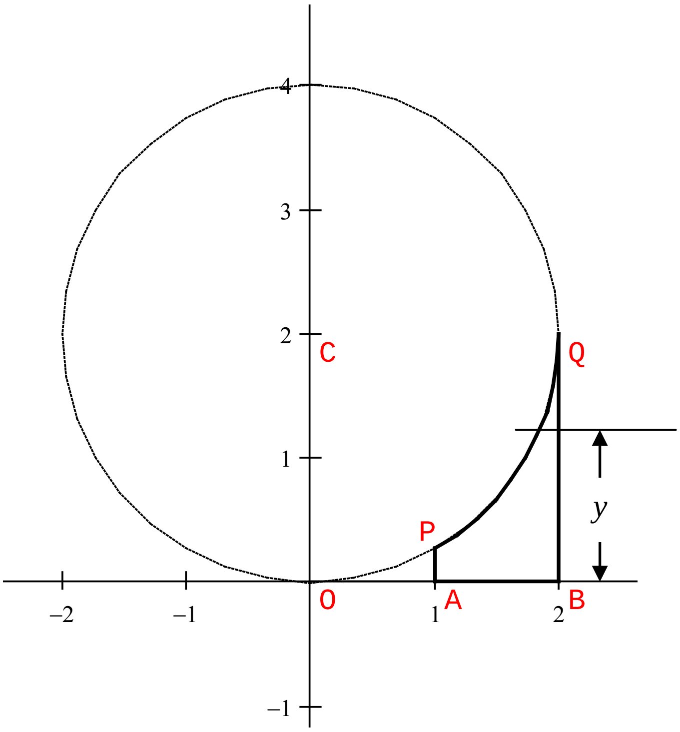
b) $P\left(a\cos\theta, b\sin\theta\right)$, where $0<\theta<\dfrac{\pi}{2}$, is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $0<b<a$. The normal to the ellipse at ***P*** has equation $ax\sin\theta - by\cos\theta = \left(a^2-b^2\right)\sin\theta\cos\theta$. This normal cuts the x-axis at ***A*** and the y-axis at ***B***.



i. Show that ΔOAB has an area given by $\dfrac{\left(a^2-b^2\right)^2}{2ab}\sin\theta\cos\theta$. **3**

ii. Find the maximum area of ΔOAB and the coordinates of ***P*** where this maximum occurs. **3**

c) In the diagram below, the shaded region is bounded by the lines $x=1$, $x=2$ the curve $x^2+(y-2)^2=4$ and the x-axis. This region is to be rotated about the y-axis. When the region is rotated, the line segment bounded on the left by the curve at height y sweeps out an annulus.



i. Show that the area of the annulus at height y is given by $\pi(y-2)^2$, where $2-\sqrt{3}\le y\le 2$. **2**

ii. Hence find the exact volume of the solid if the entire region is rotated about the y-axis, given that the cylindrical pipe portion of the solid has a volume of $\pi\left(6-3\sqrt{3}\right)$. **2**

d) The complex number $\frac{\sqrt{3}}{2}+\frac{i}{2}$ is one of the n^{th} roots of -1. Find the least value of n for this to be so. **2**

Question 8: **Marks**

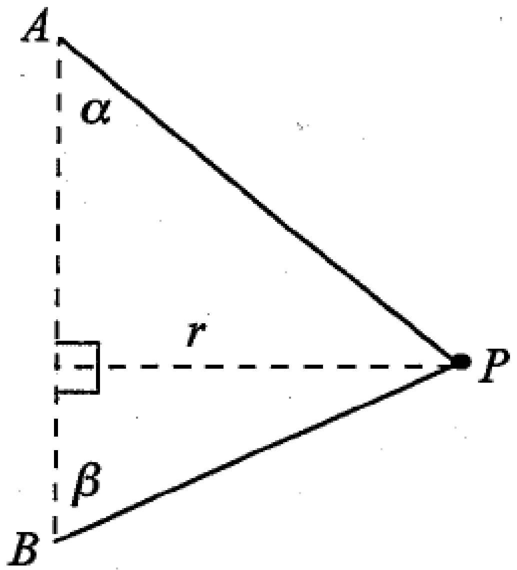
a) A straight line is drawn through a fixed point $P(a,b)$ in the first quadrant of the number plane. The line cuts the positive x-axis at ***A*** and the positive y-axis at ***B***. Given $\angle OAB = \theta$:

i. Prove that the length of ***AB*** is given by $AB = a\sec\theta + b\operatorname{cosec}\theta$. **2**

ii. Show that the length ***AB*** will be a minimum if $\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$. **3**

iii. Show that the minimum length of ***AB*** is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **2**

b) ***A*** and ***B*** are two fixed points with ***B*** vertically below ***A***. ***P*** is a particle with mass ***M*** kg. Two strings with ends fixed at ***A*** and ***B*** are fastened to ***P***. Particle ***P*** moves in a horizontal circle of radius ***r*** metres with a constant angular velocity of ω radians per second so that both strings remain taut, making angles of α, β respectively with the vertical. The tension in the strings ***AP*** and ***BP*** are T_1 Newtons and T_2 Newtons respectively. The acceleration due to gravity is ***g*** ms^{-2}.



i. Draw a diagram showing the forces acting on particle ***P***. **1**

ii. By resolving forces, show that $T_1\cos\alpha - T_2\cos\beta = Mg$ and $T_1\sin\alpha + T_2\sin\beta = Mr\omega^2$. **3**

iii. Hence show that $T_2 = \dfrac{M\left(r\omega^2\cos\alpha - g\sin\alpha\right)}{\sin\left(\alpha+\beta\right)}$. Find the corresponding expression for T_1. **2**

iv. Find the smallest possible value of ω for the motion to continue as described. Explain what happens if ω drops below this value. **2**

Question 1:

Question 1 was generally well done by most students.

a) $\int \frac{e^{2x}-1}{e^x-1}dx$

$= \int \frac{(e^x-1)(e^x+1)}{(e^x-1)}dx$

$= \int (e^x+1)dx$ ❶

$= e^x+x+c$

b) $\int \frac{\tan x}{\tan 2x}dx$

$= \int \tan x\left(\frac{1-\tan^2 x}{2\tan x}\right)dx$ ❶

$= \int \left(\frac{1-\tan^2 x}{2}\right)dx$

$= \int \frac{1}{2}\left(1-\left(\sec^2 x-1\right)\right)dx$

$= \int \frac{1}{2}\left(2-\sec^2 x\right)dx$ ❶

$= x-\frac{1}{2}\tan x+c$

c) Let $u=\sqrt{x-1}$, then $du=\frac{dx}{2\sqrt{x-1}}$; $\frac{1}{x}=\frac{1}{u^2+1}$, ❶

with limits $x=4$ $\quad x=2$ ❶

$u=\sqrt{4-1}$ $\quad u=\sqrt{2-1}$

$=\sqrt{3}$ $\quad =1$

and hence $\int_2^4 \frac{dx}{x\sqrt{x-1}}$

$= \int_1^{\sqrt{3}} \frac{du}{u^2+1}$

$= \left[\tan^{1} u\right]_1^{\sqrt{3}}$ ❶

$= \frac{\pi}{6}$

d) Let $\frac{x^2+5x-4}{(x-1)(x^2+1)} \equiv \frac{a}{(x-1)}+\frac{bx+c}{(x^2+1)}$

$x^2+5x-4=a(x^2+1)+(bx+c)(x-1)$ ❶

$x=1: 2=2a \Rightarrow a=1$ ❶

Coefficient of x^2: $1=a+b \Rightarrow b=0$

Constant: $-4=a-c \Rightarrow c=5$ ❶

Hence $\int \frac{x^2+5x-4}{(x-1)(x^2+1)}dx$

$= \int \frac{1}{x-1} + \frac{5}{x^2+1}dx$

$= \ln|x-1| + 5\tan^{-1}x + c$ ❶

e) i) Let $u = x^n$ $dv = e^{-x}$ ❶, hence

$du = nx^{n-1}$ $v = -e^{x}$

$I_n = \int_0^1 x^n e^{-x} dx$

$= \left[-e^{-x}x^n\right]_0^1 - \int_0^1 nx^{n-1}\left(-e^{-x}\right)dx$ ❶

$= -e^{-1} + nI_{n-1}$

$= nI_{n-1} - e^{-1}$

ii) From i. Above:

$I_3 = 3I_2 - e^{-1}$

$I_2 = 2I_1 - e^{-1}$

$I_1 = I_0 - e^{-1}$ ❶, and

$I_0 = \int_0^1 e^{-x} dx$

$= \left[-e^{-x}\right]_0^1$

$= 1 - e^{-1}$ ❶

Hence $I_3 = 3I_2 - e^{-1}$

$= 3\left(2I_1 - e^{-1}\right) - e^{-1}$

$= 3\left(2\left(I_0 - e^{-1}\right) - e^{-1}\right) - e^{-1}$

$= 3\left(2\left(1 - e^{-1} - e^{-1}\right) - e^{-1}\right) - e^{-1}$ ❶

$= 6 - 16e^{-1}$

Question 2:

a) i) $|z|=\sqrt{(\sqrt{3})^2+1^2}$ $\arg(z)=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ ❶

$=\sqrt{4}$ $=\frac{-\pi}{6}$

$=2$

Hence $z=2cis\left(\frac{-\pi}{6}\right)$

$=2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$ ❶

ii) $(\sqrt{3}-i)^6=\left(2cis\frac{-\pi}{6}\right)^6$

$=2^6\times cis\frac{-6\pi}{6}$ ❶

$=64cis(-\pi)$

$=64\times-1$

$=-64$ ❶

Well done.

cis notation is an abbreviation that should not be used for the final answer.

b) i) Substituting $z=1+i$ into $z^2-aiz+b=0$:

$(1+i)^2-ai(1+i)+b=0$

$1+2i-1-ai+a+b=0$

$(a+b)+i(2-a)=0$ ❶

Equating real and imaginary parts:

$2-a=0\Rightarrow a=2$

$a+b=0\Rightarrow b=-2$ ❶

ii) Let the second root be β.

Then $\sum\alpha\beta=\frac{-b}{a}$ gives

$(1+i)+\beta=2i$ ❶

$\beta=-1+i$ ❶

Note: co-efficients are not real, so the conjugate is not a solution!

c) $z=r(\cos\theta+i\sin\theta)$

$z^2=r^2(\cos 2\theta+i\sin 2\theta)$ ❶

$z^2+r^2=r^2(1+\cos 2\theta+i\sin 2\theta)$

$=r^2(2\cos^2\theta+2i\sin\theta\cos\theta)$

$=2r^2\cos\theta(\cos\theta+i\sin\theta)$ ❶

$=2r\cos\theta.z$

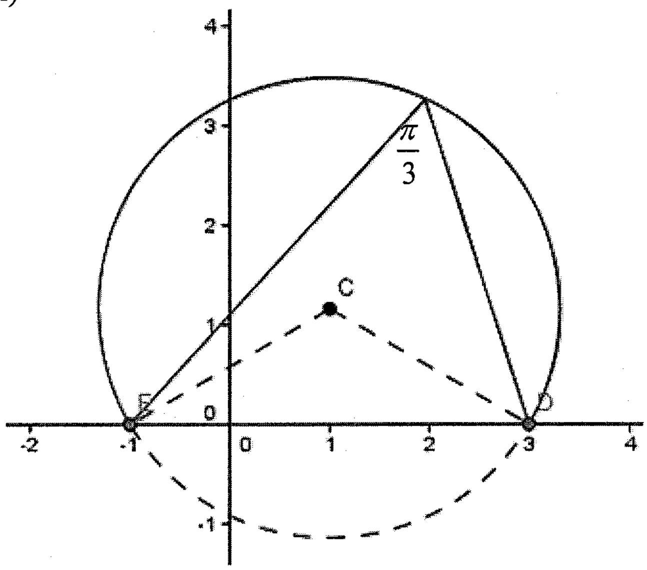
$\frac{1}{z^2+r^2}=\frac{1}{2r\cos\theta.z}$

$\frac{z}{z^2+r^2}=\frac{z}{2r\cos\theta.z}$, hence

Many algebraic errors in this part.

$$\frac{z}{z^2+r^2}=\frac{1}{2r\cos\theta}$$ ❶ (which is real)

d) i)



correct interpretation of arg: ❶, correct arc: ❶

Mostly well done.

ii) Angle at the centre: $\angle ECD=\frac{2\pi}{3}$,

so $\angle CDE=\dfrac{\pi-\frac{2\pi}{3}}{2}$

$=\frac{\pi}{6}$

$\tan\frac{\pi}{6}=\frac{y_C}{2}$

$y_C=2\tan\frac{\pi}{6}$

$=\frac{2}{\sqrt{3}}$ and $x_C=1$ (midpt of ED)

Thus Centre is $\left(1,\frac{2}{\sqrt{3}}\right)$ ❶, then

$\cos\frac{\pi}{6}=\frac{2}{r}$

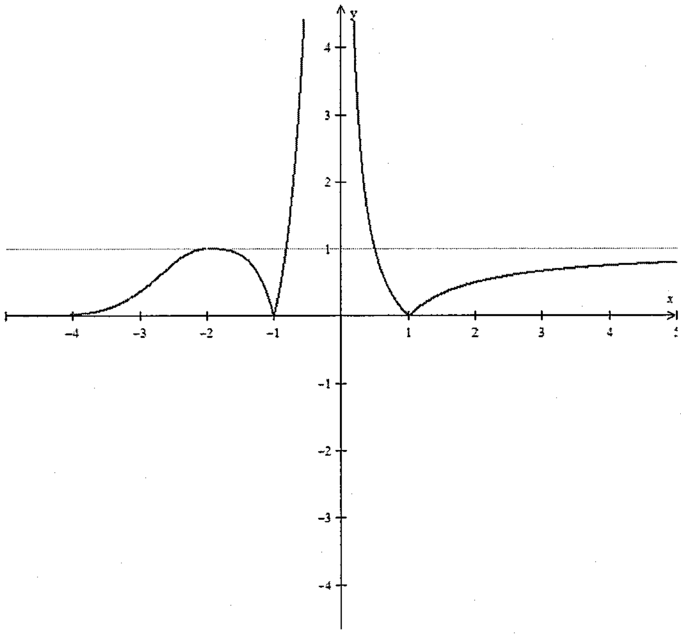
$r=\dfrac{2}{\cos\frac{\pi}{6}}$

$r=\frac{4}{\sqrt{3}}$ ❶

Question 3:

a)

i) $y=|f(x)|$



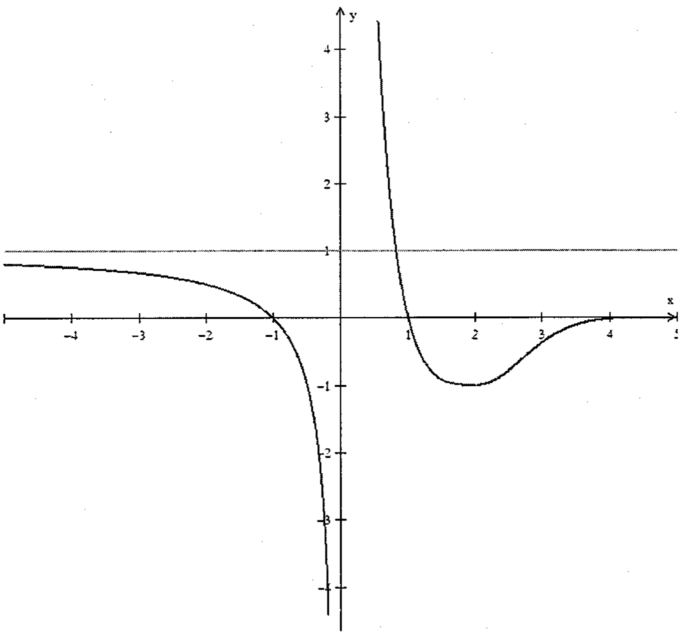
➊ graph correct

Many mis-interpreted the graph as being $y=0$ for $x\le -4$. Those who were consistent in this interpretation were not penalised.

Note: in all graphs, *y*-values of 0 and 1 are very important – many did not interpret these correctly!

$x=\pm 1$ should be sharp cusps for absolute value!

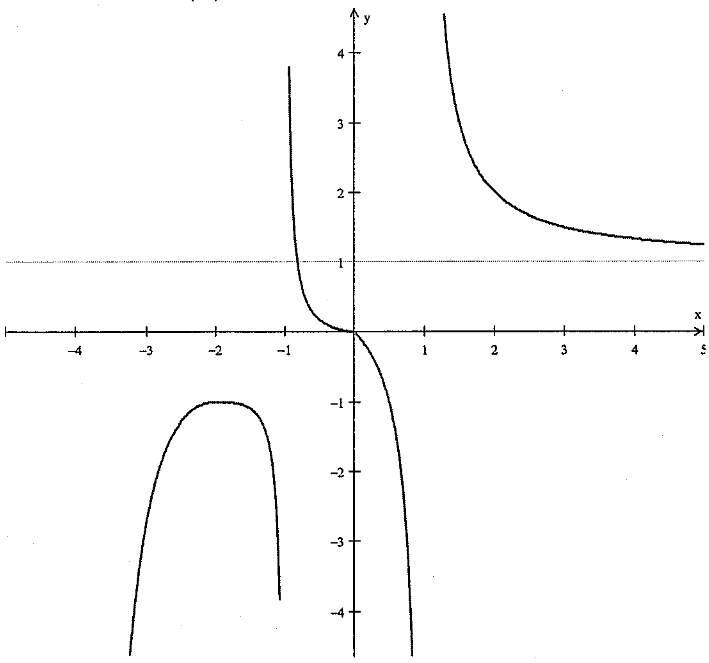
ii) $y=f(-x)$



➊ graph correct

This is a reflection in the *y*-axis - some only reflected half the graph

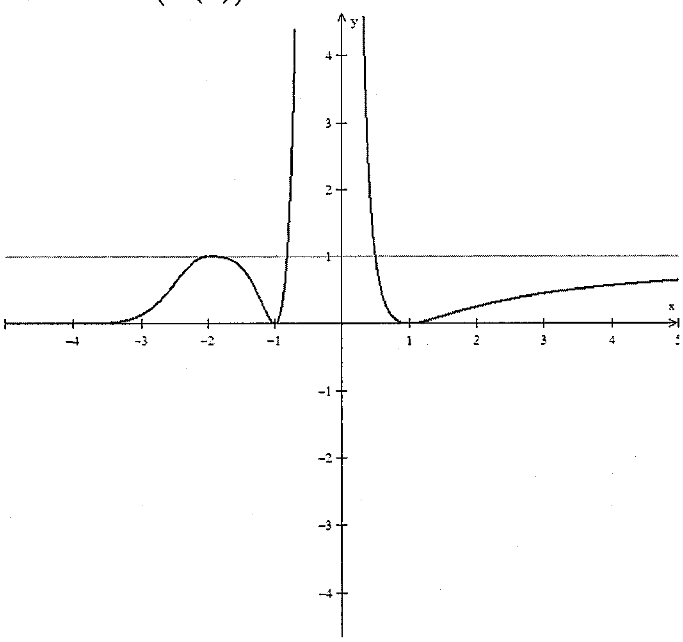
iii) $y = \dfrac{1}{f(x)}$



Asymptotes need to be clearly indicated.

➊ limit as $x \to \infty$ and tp at $x = -2$, ➊ graph shape correct

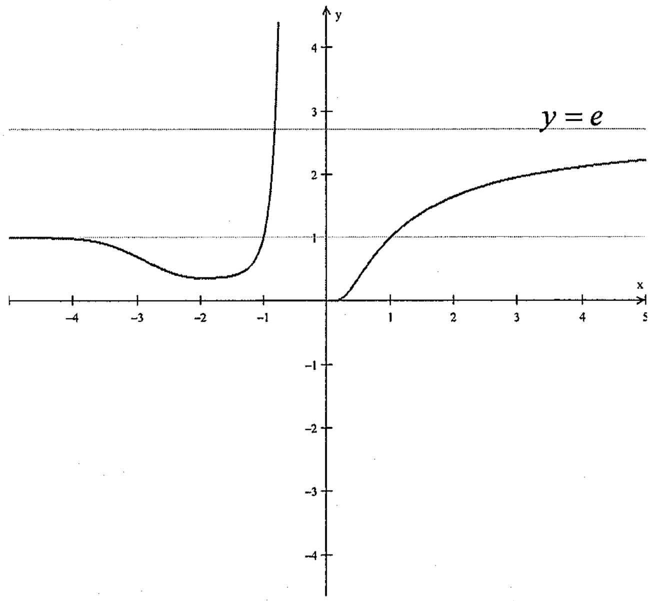
iv) $y = (f(x))^2$



Squared values at $x = \pm 1$ should look parabolic, not cusp-like.

➊ shape at x=-1, x=0 and x=1; ➊ graph shape correct

v) $y = e^{f(x)}$



As $x \to 0$, the graph approaches zero!

❶ asymptotes as $x \to \pm\infty$; ❶ graph shape correct

b) i) In the first quadrant: $\sqrt{x} + \sqrt{y} = \sqrt{L}$, so

$$\sqrt{y} = \sqrt{L} - \sqrt{x}$$

$$y = \left(\sqrt{L} - \sqrt{x}\right)^2$$

$$= L + x - 2\sqrt{Lx} \text{ ❶, hence}$$

$$A = \int_0^L L + x - 2\sqrt{Lx}\,dx$$

$$= \left[Lx + \frac{1}{2}x^2 - \frac{2}{3}.2\sqrt{L}.x^{\frac{3}{2}} \right]_0^L$$

$$= \left(L^2 + \frac{1}{2}L^2 - \frac{4\sqrt{L}}{3}.L\sqrt{L} \right) - 0$$

$$= \frac{L^2}{6} \text{ ❶}$$

Many did not expand this and thus did not integrate correctly

Thus, over the 4 quadrants,

$$A = 4 \times \frac{L^2}{6} \text{ ❶}$$

$$= \frac{2}{3}L^2 \quad \text{as reqd.}$$

Many were unclear with the symmetry, and used 2 instead of 4 quadrants.

ii) At height h: $\delta A = \frac{2}{3}l^2$, with $l = L\left(1 - \frac{h}{H}\right)$

Thus $\delta A = \frac{2}{3}L^2\left(1-\frac{h}{H}\right)^2$

$= \frac{2}{3}L^2\left(1-\frac{2h}{H}+\frac{h^2}{H^2}\right)$ ❶

Hence $\delta V = \delta A.\delta h$

$= \frac{2}{3}L^2\left(1-\frac{2h}{H}+\frac{h^2}{H^2}\right)\delta h$ ❶

Now $V = \lim_{h\to 0}\sum_{h=0}^{H}\delta V$

$= \lim_{h\to 0}\sum_{h=0}^{H}\frac{2}{3}L^2\left(1-\frac{2h}{H}+\frac{h^2}{H^2}\right)\delta h$ ❶

$= \int_0^H \frac{2}{3}L^2\left(1-\frac{2h}{H}+\frac{h^2}{H^2}\right)dh$

So $V = \frac{2}{3}L^2\left[h-\frac{h^2}{H}+\frac{h^3}{3H^2}\right]_0^H$

$= \frac{2}{3}L^2\left(H-\frac{H^2}{H}+\frac{H^3}{3H^2}\right)-0$

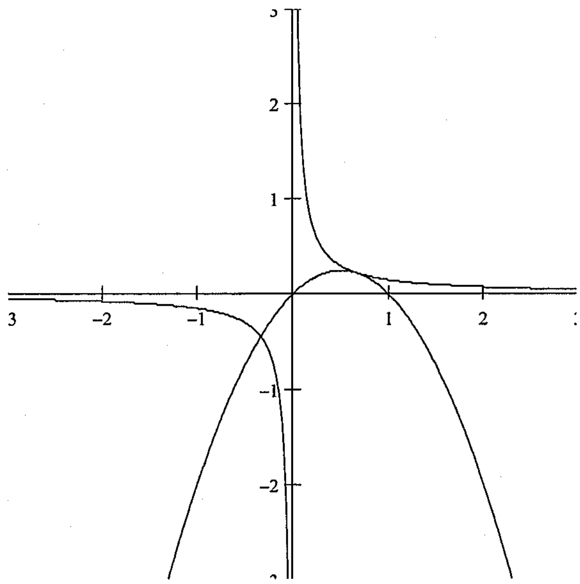
$= \frac{2L^2H}{9}$ ❶

Many did not develop the sequence of
i) finding δA
ii) finding δV
iii) showing
$V = \lim\sum\delta V$
which is the standard requirement for this style of question!

Question 4:

a)

i)



❶ graph correct

Must show tangential at Q, intersecting at R for 1 mark

ii) Solving simultaneously:

$$x\left(x-x^2\right)=c^2$$

$$x^2-x^3=c^2$$

$$x^3-x^2+c^2=0$$

Considering $P(x)=x^3-x^2+c^2$

$$P'(x)=3x^2-2x$$

Now when $P'(x)=0$

$$0=3x^2-2x$$

$$=x(3x-2)\text{ ❶}$$

Thus $x=0$ $\quad x=\frac{2}{3}$ are possible multiple roots

As the curves are tangential, and $x\neq 0$ for the hyperbola, there must be adouble root at $x=\frac{2}{3}$ ❶, hence $P\left(\frac{2}{3}\right)=0$, so

Must be clear why $x=\frac{2}{3}$ gives the double root, not $x=0$

$$0=\left(\frac{2}{3}\right)^3-\left(\frac{2}{3}\right)^2+c^2$$

$$=\frac{8}{27}-\frac{4}{9}+c^2$$

$$c^2=\frac{4}{27}\text{ ❶}$$

iii)

$P(x) = x^3 - x^2 + \frac{4}{27}$, but also

$P(x) = \left(x - \frac{2}{3}\right)^2 (x - b)$ from (ii), hence

$\frac{4}{27} = \left(\frac{-2}{3}\right)^2 \times (-b)$ or $b = \frac{-1}{3}$; with $\frac{-1}{3}.y = \frac{4}{27}$

the coordinates of R are therefore $\left(\frac{-1}{3}, \frac{-4}{9}\right)$ ❶.

Exact coordinates required

b)

i) Differentiating:

$$\frac{2x}{a^2} - \frac{2y}{b^2}.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{a^2}.\frac{b^2}{-2y}$$

$$= \frac{b^2 x}{a^2 y}$$ ❶

At $P(a\sec\theta, b\tan\theta)$; $\frac{dy}{dx} = \frac{b^2 a\sec\theta}{a^2 b\tan\theta}$

$$= \frac{b\sec\theta}{a\tan\theta}$$ ❶

Hence equation of normal is:

$$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$$ ❶

$$by\sec\theta - b^2\sec\theta\tan\theta = -ax\tan\theta + a^2\sec\theta\tan\theta$$

$$ax\tan\theta + by\sec\theta = a^2\sec\theta\tan\theta + b^2\sec\theta\tan\theta$$

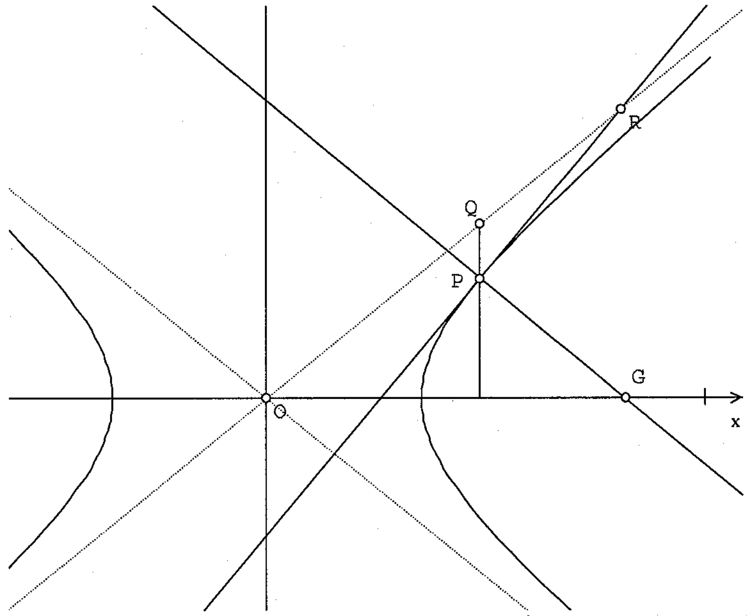
$$= \sec\theta\tan\theta\left(a^2 + b^2\right)$$ ❶

Dividing by $\sec\theta\tan\theta$ gives

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$ as reqd.

Most did this part very well

ii)



The normal at P crosses the x-axis at $G\left(\dfrac{\left(a^2+b^2\right)\sec\theta}{a},0\right)$.

Q has the same x-coordinate as P and is on the asymptote $y=\dfrac{b}{a}x$, so has coordinates $\left(a\sec\theta, b\sec\theta\right)$.

Hence the gradient of QR is $\dfrac{b}{a}$ (both on the asymptote $y=\dfrac{b}{a}x$).

The gradient of QG is:

$$\frac{b\sec\theta-0}{a\sec\theta-\dfrac{\left(a^2+b^2\right)\sec\theta}{a}}$$

$$=\frac{ab\sec\theta}{a^2\sec\theta-\left(a^2+b^2\right)\sec\theta}$$

$$=\frac{ab\sec\theta}{a^2\sec\theta-a^2\sec\theta-b^2\sec\theta}$$

$$=\frac{ab\sec\theta}{-b^2\sec\theta}$$

$$=-\frac{a}{b}$$ ❶

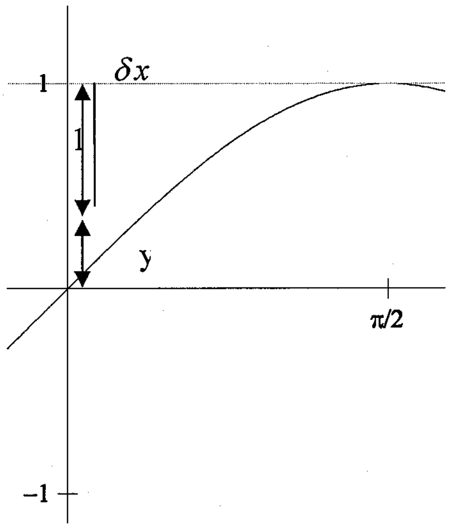
Thus $m_{QR}\times m_{QG}=\dfrac{b}{a}\times\dfrac{-a}{b}$

$=-1$ ❶

So $QR\perp QG$ hence $\angle RQG=\dfrac{\pi}{2}$.

Full marks for obtaining both gradients QR, QG. Some students did not give an explanation for $m_{QR}=\dfrac{b}{a}$

c)



Area of slice rotating about $y=1$:

$\delta A = \pi(1-y)^2$

$= \pi(1-\sin x)^2$ ❶

Volume of each slice:

$\delta V = \pi(1-\sin x)^2\,\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{\frac{\pi}{2}} \delta V$

$= \pi \int_0^{\frac{\pi}{2}} (1-\sin x)^2\,dx$

$= \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \sin^2 x\,dx$

$= \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$ ❶

$= \pi\left[\frac{3x}{2} + 2\cos x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$ ❶

$= \pi\left[\left(\frac{3\pi}{4} + 0 + 0\right) - (0 + 2 - 0)\right]$

$= \frac{\pi(3\pi - 8)}{4}$ ❶ *cubic units*

A common error was to use

$\delta A = \pi - \pi(1 - y^2)$

3 marks were generally awarded in this case if other working accurate.

Question 5:

a)

i) $\cos 4\theta = \frac{1}{2}$

$$4\theta = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$$

$$= 2n\pi \pm \frac{\pi}{3} \text{ ❶}$$

$$= \left(\frac{6n \pm 1}{3}\right)\pi$$

$$\theta = \left(\frac{6n \pm 1}{12}\right)\pi, \quad n = 0, \pm 1, \pm 2 \ldots \text{ ❶}$$

i) Most students lost marks because they did not state what *n* is equal to.

Without restriction *n* is interpreted as any number and therefore incorrect.

ii) $cis^4\theta = (\cos\theta + i\sin\theta)^4$

$cis^4\theta = cis4\theta$ by DeMoivre, and

$(\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= \left(\cos^4\theta + 6\cos^2\theta(i\sin\theta)^2 + (i\sin\theta)^4\right) + i\left(4\cos^3\theta(\sin\theta) - 4\cos\theta(\sin\theta)^3\right) \text{ ❶}$$

Equating real parts:

$$\cos 4\theta = \cos^4\theta + 6\cos^2\theta(i\sin\theta)^2 + (i\sin\theta)^4 \text{ ❶}$$

$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \text{ ❶}$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

ii) The question asked specifically to demonstrate De Moivre's theorem and students shoud quote this when writing $cis^4\theta = cis4\theta$

Many students lost marks because they skipped steps. *Show that* questions require students to explicitly demonstrate knowledge of identities and the ability to expand and simplify expressions.

iii) $16x^4 - 16x^2 + 1 = 0$

Let $x = \cos\theta$, then the equation becomes

$$16\cos^4\theta - 16\cos^2\theta + 1 = 0$$

$$2\left(8\cos^4\theta - 8\cos^2\theta + 1\right) - 1 = 0$$

$$8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$$

$$\cos 4\theta = \frac{1}{2} \text{ ❶}$$

Then from (i), we get the following:

iii) Some students had difficulty in communicating the relationship between $16x^4 - 16x^2 + 1 = 0$ and $\cos 4\theta = \frac{1}{2}$.

$n=0:\theta=\dfrac{0\pm1\pi}{12}\qquad \therefore\theta=\dfrac{-\pi}{12}\,or\,\dfrac{\pi}{12}$

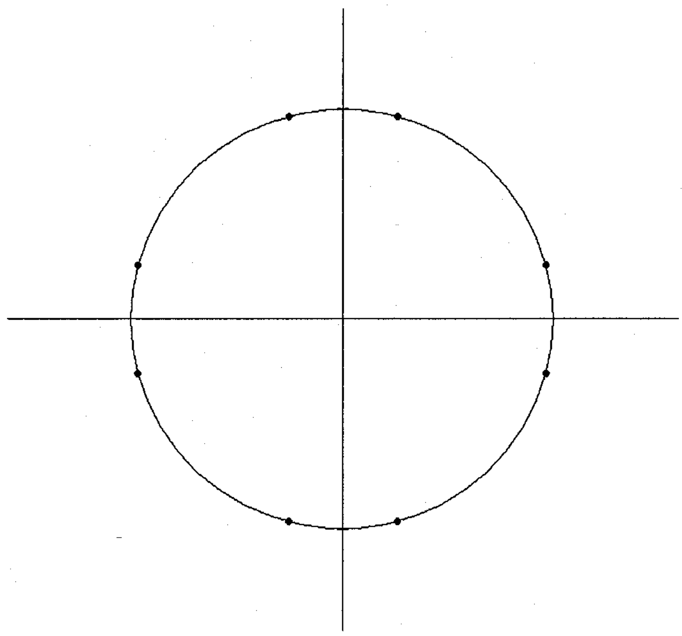
$n=1:\theta=\dfrac{(6\pm1)\pi}{12}\qquad \therefore\theta=\dfrac{5\pi}{12}\,or\,\dfrac{7\pi}{12}$

$n=2:\theta=\dfrac{(12\pm1)\pi}{12}\qquad \therefore\theta=\dfrac{11\pi}{12}\,or\,\dfrac{13\pi}{12}$

$n=3:\theta=\dfrac{(18\pm1)\pi}{12}\qquad \therefore\theta=\dfrac{17\pi}{12}\,or\,\dfrac{19\pi}{12}$

$n=4:\theta=\dfrac{(24\pm1)\pi}{12}\qquad \therefore\theta=\dfrac{23\pi}{12}\,or\,\dfrac{25\pi}{12}$

Thus:



Resolving for distinct roots and with $0\le\theta\le2\pi$: ❶

$x_1=\cos\dfrac{\pi}{12}\left(=\cos\dfrac{23\pi}{12}\right)$

$x_2=\cos\dfrac{5\pi}{12}\left(=\cos\dfrac{19\pi}{12}\right)$

$x_3=\cos\dfrac{7\pi}{12}\left(=\cos\dfrac{17\pi}{12}\right)$

$x_4=\cos\dfrac{11\pi}{12}\left(=\cos\dfrac{13\pi}{12}\right)$

iv) Let $u=x^2$, the equation becomes

$$0=16u^2-16u+1$$

$$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{16\pm\sqrt{256-4\times16\times1}}{2\times16}$$

$$=\frac{16\pm\sqrt{192}}{32}$$

$$=\frac{16\pm8\sqrt{3}}{32}$$

$$=\frac{2\pm\sqrt{3}}{4}\ ❶$$

Then

$$x^2=\frac{2\pm\sqrt{3}}{4}$$

$$x=\pm\frac{\sqrt{2\pm\sqrt{3}}}{2}$$

But $\cos\frac{\pi}{2}<\cos\frac{5\pi}{12}<\cos\frac{\pi}{12}<\cos 0$

So $\cos\frac{\pi}{12}$ is the biggest positive value for x above ❶,

Hence $\cos\frac{\pi}{12}=\frac{\sqrt{2+\sqrt{3}}}{2}$, as reqd.

Many students lost marks because they could not explain why they had to disregard some solutions when $x=-\frac{\sqrt{2\pm\sqrt{3}}}{2}$ (some did not even acknowledge these solution exists)

And why when taking the positive case the answer was $\cos\frac{\pi}{12}$ and not $\cos\frac{5\pi}{12}$.

b)

i) $\frac{dv}{dt}=-kv^3$

Hence $\frac{dv}{v^3}=-kdt$

$$-kt=-\frac{1}{2v^2}+c$$

$$t=\frac{1}{2kv^2}+c$$

Initially, $t=0, v=U$:

$$0=\frac{1}{2kU^2}+c$$

$$c=\frac{-1}{2kU^2}\ ❶$$

Hence $t=\frac{1}{2kv^2}-\frac{1}{2kU^2}$

$$=\frac{1}{2k}\left(\frac{1}{v^2}-\frac{1}{U^2}\right)$$

At $t=T$:

$$T = \frac{1}{2k}\left(\frac{1}{v^2} - \frac{1}{U^2}\right)$$ ❶

$$\left(\frac{1}{v^2} - \frac{1}{U^2}\right) = 2kT$$, as reqd.

ii) $v\frac{dv}{dx} = -kv^3$

$\frac{v}{v^3}dv = -kdx$

$-kdx = \frac{dv}{v^2}$

$-kx + c = \frac{-1}{v}$ ❶

Initially, $x = 0, v = U$:

$c = \frac{-1}{U}$, hence

$-kx + \frac{-1}{U} = \frac{-1}{v}$

$kx = \frac{1}{v} - \frac{1}{U}$ ❶

When $x = D, v = V$:

$kD = \frac{1}{V} - \frac{1}{U}$, as reqd.

Students lost marks because they could not adequately show how they derived the answer.

Question 6: Marks

a) $\alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\alpha\gamma=k$, $\alpha\beta\gamma=-2$

i) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$ ❶

$=-\frac{k}{2}$ ❶

ii) With $\alpha,\beta\,\&\,\gamma$ roots of $x^3+kx+2=0$:

Re-arranging this to $x^3=-kx-2$, and substituting the roots above gives

$\alpha^3=-k\alpha-2$

$\beta^3=-k\beta-2$

$\gamma^3=-k\gamma-2$, then adding we get:

$\alpha^3+\beta^3+\gamma^3=-k(\alpha+\beta+\gamma)-6$ ❶

$=-k\times(0)-6$

$=-6$ ❶

which in independent of k as reqd.

iii) $x^3+kx+2=0$ with roots $\alpha,\beta\,\&\,\gamma$

New roots: $y=\alpha^2,\beta^2,\gamma^2 \Rightarrow y=x^2 \text{ or } x=\sqrt{y}$. Thus

$\left(\sqrt{y}\right)^3+k\sqrt{y}+2=0$

$y\sqrt{y}+k\sqrt{y}=-2$

$\sqrt{y}\left(y+k\right)=-2$ ❶

Squaring both sides:

$y\left(y+k\right)^2=4$ ❶

$y\left(y^2+2ky+k^2\right)=4$

$y^3+2ky^2+k^2y=4$

Hence the monic equation with roots $\alpha^2,\beta^2\,\&\,\gamma^2$ is

$x^3+2kx^2+k^2x-4=0$ ❶

Students who arrived at incorrect answers typically tried to find a relationship between $\alpha^3+\beta^3+\gamma^3$ and $(\alpha+\beta+\gamma)^3$ rather than the method illustrated.

Students also have to state/show that $\alpha+\beta+\gamma=0$

Students should also show that they know -6 is independent of k by a concluding statement (i.e. -6 is independent of k)

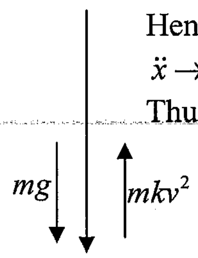
Students did not receive full marks unless they provided an equation with integer powers.

b)

i) Resultant $F=\sum ma$, so $m\ddot{x}=mg-mkv^2$

Hence $\ddot{x}=g-kv^2$ ❶, and terminal velocity occurs as $\ddot{x}\to 0$

Thus $0=g-kV_t^2$

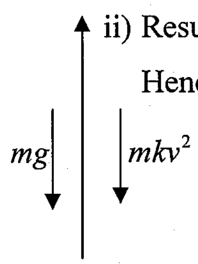


$V_t^2=\frac{g}{k}$

$V_t=\sqrt{\frac{g}{k}}$ ❶

ii) Resultant $F=\sum ma$, so $m\ddot{x}=-mg-mkv^2$

Hence $\ddot{x}=-\left(g+kv^2\right)$



Relating t with v:

$$\frac{dv}{dt} = -\left(g + kv^2\right)$$

$$-dt = \frac{dv}{\left(g + kv^2\right)}$$

$$= \frac{dv}{k\left(\frac{g}{k} + v^2\right)}$$

$$-kdt = \frac{dv}{\left(\frac{g}{k} + v^2\right)}$$

Integrating:

$$-kt + c = \sqrt{\frac{k}{g}} \tan^{-1}\left(\sqrt{\frac{k}{g}} v\right)$$ ❶

Initially, $t = 0, v = U$

$$c = \sqrt{\frac{k}{g}} \tan^{-1}\left(\sqrt{\frac{k}{g}} U\right)$$, thus

$$kt = \sqrt{\frac{k}{g}} \tan^{-1}\left(\sqrt{\frac{k}{g}} U\right) - \sqrt{\frac{k}{g}} \tan^{-1}\left(\sqrt{\frac{k}{g}} v\right)$$

At max height, $t = T, v = 0$

$$kT = \sqrt{\frac{k}{g}} \tan^{-1}\left(\sqrt{\frac{k}{g}} U\right) - 0$$

$$T = \frac{1}{k}\sqrt{\frac{k}{g}} \tan^{-1}\left(\sqrt{\frac{k}{g}} U\right)$$ ❶

And from (i) $V_t = \sqrt{\frac{g}{k}} \Rightarrow \sqrt{\frac{k}{g}} = \frac{1}{V_t}$, thus

$$T = \frac{1}{k}\frac{1}{V_t} \tan^{-1}\left(\frac{U}{V_t}\right)$$

Also from (i): $V_t^2 = \frac{g}{k} \Rightarrow k = \frac{g}{V_t^2}$, hence

$$T = \frac{1}{\frac{g}{V_t^2}}\frac{1}{V_t} \tan^{-1}\left(\frac{U}{V_t}\right)$$ ❶

$$T = \frac{V_t}{g} \tan^{-1}\left(\frac{U}{V_t}\right)$$, as reqd.

Students did not show enough working in substituting terminal velocity into the equation.

iii) Relating x with v

$$v\frac{dv}{dx} = -\left(g + kv^2\right)$$

$$-dx = \frac{v\,dv}{\left(g + kv^2\right)}$$

$$-2k\,dx = \frac{2kv\,dv}{\left(g + kv^2\right)}$$

Integrating:

$$-2kx = \ln\left|g + kv^2\right| + c \text{ ❶}$$

Initially, $x = 0, v = U$ gives:

$c = -\ln\left|g + kU^2\right|$, thus

$$-2kx = \ln\left|g + kv^2\right| - \ln\left|g + kU^2\right|$$

$$x = \frac{1}{2k}\left(\ln\left|g + kU^2\right| - \ln\left|g + kv^2\right|\right)$$

$$= \frac{1}{2k}\ln\left|\frac{g + kU^2}{g + kv^2}\right|$$

At max height: $x = H, v = 0$

$$H = \frac{1}{2k}\ln\left|\frac{g + kU^2}{g}\right|$$

$$= \frac{1}{2k}\ln\left|1 + \frac{kU^2}{g}\right| \text{ ❶}$$

Again, from (i), with $\frac{k}{g} = \frac{1}{V_t^2}; k = \frac{g}{V_t^2}$:

$$H = \frac{V_t^2}{2g}\ln\left|1 + \frac{U^2}{V_t^2}\right| \text{ ❶}$$

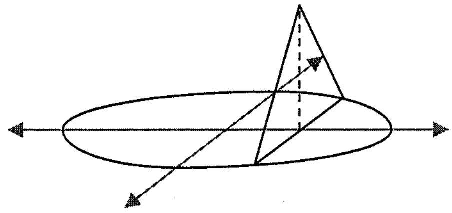
$$= \frac{V_t^2}{2g}\ln\left|\frac{V_t^2 + U^2}{V_t^2}\right|, \text{ as reqd.}$$

Students should be aware that there are many different ways you could show your working and that care should be taken to ensure examiners are able to read and logically follow your solution.

Question 7: Marks

a) Semi-major axis is 5 units, semi-minor axis is 4 units:

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \Rightarrow y=4\sqrt{1-\frac{x^2}{25}}$$



Cross-sectional Area: $\delta A=\frac{1}{2}bh$

$$=\frac{1}{2}(2y).6$$

$$=6y$$

Slice of volume: $\delta V=6y.\delta x$

$\therefore$ Volume is $V=\lim_{\delta x\to 0}\sum_{-5}^{5}\delta V$

$$=\lim_{\delta x\to 0}\sum_{-5}^{5}6y\delta x \text{ ❶}$$

Taking limits:

$$V=\int_{-5}^{5}6y\,dx$$

$$=6\int_{-5}^{5}4\sqrt{1-\frac{x^2}{25}}\,dx$$

$$=24\int_{-5}^{5}\sqrt{\frac{25-x^2}{25}}\,dx$$

$$=\frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}\,dx \text{ ❶}$$

Then noting $\int_{-5}^{5}\sqrt{25-x^2}\,dx$ is the area of a semi-circle with $r=5$:

$$V=\frac{24}{5}.\frac{1}{2}\pi r^2$$

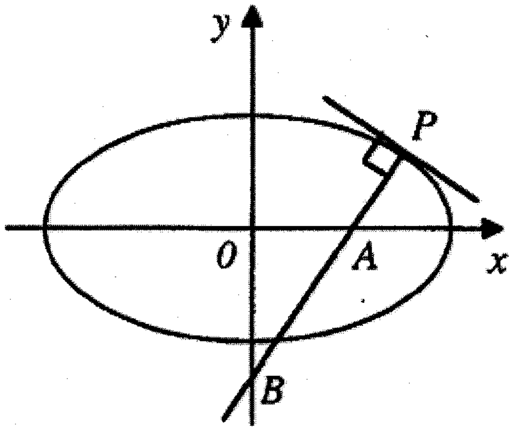
$$=\frac{24}{10}\pi(25)$$

$$=60\pi\ c.u. \text{ ❶}$$

Common errors were to use the equation $\frac{x^2}{100}+\frac{y^2}{64}=1$ and have the wrong terminals for the integration.

Some students used substitution as a method of integration and were marked accordingly.

b)



i. At A: $y=0 \Rightarrow x=\dfrac{a^2-b^2}{a}\cos\theta$ ❶

At B: $x=0 \Rightarrow y=-\dfrac{a^2-b^2}{b}\sin\theta$ ❶, hence Area:

$$\Delta AOB=\frac{1}{2}.OA.OB$$
$$=\frac{1}{2}.\frac{a^2-b^2}{a}\cos\theta.\frac{a^2-b^2}{b}\sin\theta \text{ ❶}$$
$$=\frac{\left(a^2-b^2\right)^2}{2ab}\sin\theta\cos\theta \quad \text{, as reqd.}$$

Generally well done.

ii. Noting $\sin\theta\cos\theta=\frac{1}{2}\sin 2\theta$, then

$\sin\theta\cos\theta$ has a maximum value of $\frac{1}{2}$ when $\theta=\frac{\pi}{4}$ ❶

Hence max area is

$$A=\frac{\left(a^2-b^2\right)^2}{2ab}.\frac{1}{2}$$
$$=\frac{\left(a^2-b^2\right)^2}{4ab} \text{ ❶}$$

$\therefore P$ has coordinates $\left(a\cos\frac{\pi}{4}, b\cos\frac{\pi}{4}\right)$ or

$\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ ❶

Some students knew that $\theta=\frac{\pi}{4}$ gave a maximum area but did not write down the maximum area.

Many students used calculus to show that $\theta=\frac{\pi}{4}$ gives the maximum area but did not check that it was a maximum by use of the second derivative or an alternative method, but no mark was deducted in this case.

c) .

i. At ***P***: $x=1$, $\Rightarrow (y-2)^2 = 3$

So $y-2=\pm\sqrt{3}$

$y = 2\pm\sqrt{3}$ ❶

But ***P*** is in the 4th quadrant of the circle, so $P\left(1, 2-\sqrt{3}\right)$

Area of Annulus: $A = \pi R^2 - \pi r^2$ where $R = 2, r = x$, thus $A = \pi\left(2^2 - x^2\right)$, and rearranging original equation:

$4 - x^2 = (y-2)^2$, so

$A = \pi\left(4 - x^2\right)$ ❶

$= \pi(y-2)^2$, as reqd.

ii. Total volume is Annulus + Cylinder:

$$V = \int_{2-\sqrt{3}}^{2} \pi(y-2)^2\,dy + \pi\left(6-3\sqrt{3}\right)$$

$$= \frac{\pi}{3}\left[(y-2)^3\right]_{2-\sqrt{3}}^{2} + \pi\left(6-3\sqrt{3}\right)$$

$$= \frac{\pi}{3}\left[0 - \left(2-\sqrt{3}-2\right)^3\right] + \pi\left(6-3\sqrt{3}\right) \text{ ❶}$$

$$= \frac{\pi}{3}.3\sqrt{3} + 6\pi - 3\pi\sqrt{3}$$

$$= \left(6-2\sqrt{3}\right)\pi \text{ ❶}$$

Students did not seem clear that they needed to show $2-\sqrt{3} \le x \le 2$.

1 mark awarded for showing that $A = \pi(y-2)^2$. Working required as this area was given in the question.

Some students used very inappropriate methods and were marked accordingly.

d) Let $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, then

$z = cis\left(\frac{\pi}{6}\right)$ in mod/arg form

If $z^n = -1$, and noting $-1 = cis(\pi)$, then

$cis^n\left(\frac{\pi}{6}\right) = cis(\pi)$ ❶

$cis\left(\frac{n\pi}{6}\right) = cis(\pi)$ by DeMoivre. Hence

$\frac{n\pi}{6} = \pi$

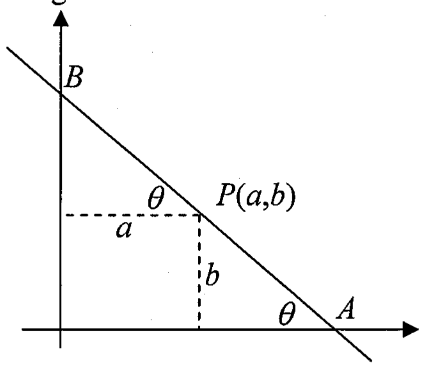
$n = 6$ ❶

Thus the least value of n for $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ is an n^{th} root of -1 if $n = 6$.

Needed to establish $cis\frac{n\pi}{6} = cis\pi$ to be awarded 1 mark.

Question 8: Marks

a) Diagram:



i. $\sin\theta = \dfrac{b}{AP}$, so

$$AP = \frac{b}{\sin\theta} = b\operatorname{cosec}\theta \ ❶$$

Similarly,

$$\cos\theta = \frac{a}{PB}$$

$$PB = \frac{a}{\cos\theta} = a\sec\theta \ ❶$$

Hence

$$AB = AP + PB = a\sec\theta + b\operatorname{cosec}\theta$$, as reqd.

Many did not place θ correctly in the diagram.

Many made this more complicated than it was with their working.

ii. Differentiating:

$$\frac{d(AB)}{d\theta} = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta$$

For a stat pt. $\dfrac{d(AB)}{d\theta} = 0$

$$0 = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta$$

$$a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$$

$$\frac{a}{b} = \frac{\cot\theta\operatorname{cosec}\theta}{\tan\theta\sec\theta} \ ❶$$

$$= \frac{\cot\theta}{\tan\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$= \cot^3\theta \ ❶$$

$$\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$$, as reqd.

Many did not <u>clearly</u> show the sequence of steps necessary in this style of question.

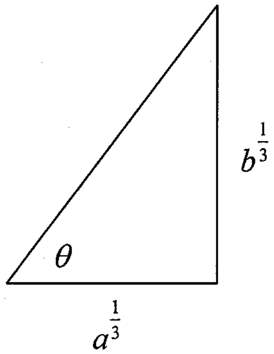
Also:

$$\frac{d^2(AB)}{d\theta^2} = a\sec^3\theta + a\sec\theta\tan^2\theta + b\operatorname{cosec}\theta\cot^2\theta + b\operatorname{cosec}^3\theta$$

Since $0 \le \theta \le \frac{\pi}{2}$, and all trig ratios are >0, then $\frac{d^2(AB)}{d\theta^2} > 0$, so this is a maximum value. ❶

Many did not show a minimum.

iii. $\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}} \Rightarrow \theta$ is acute.



DRAW A DIAGRAM!

Hence $h^2 = \left(a^{\frac{1}{3}}\right)^2 + \left(b^{\frac{1}{3}}\right)^2$

$$= a^{\frac{2}{3}} + b^{\frac{2}{3}}$$

$$h = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

Thus

$$\sec\theta = \frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}}; \operatorname{cosec}\theta = \frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}} \text{ ❶}$$

Again, many made this more complex than it needed to be.

$\therefore$ minimum length of AB is:
$AB = a\sec\theta + b\operatorname{cosec}\theta$

$$= a\frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}} + b\frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$$

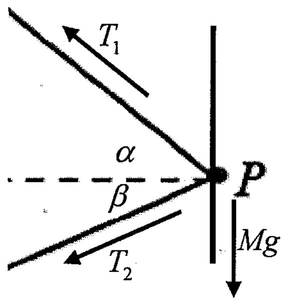
$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot \left(\frac{a}{a^{\frac{1}{3}}} + \frac{b}{b^{\frac{1}{3}}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right) \text{ ❶}$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad \text{, as reqd}$$

b)

i.



❶ (diagram correct)

Well done

ii. Since ***P*** is performing uniform circular motion horizontally, by Newton's 2nd Law of Motion, the resultant force is $mr\omega^2$ towards the centre of the circle ❶; vertically, the resultant force is zero (as there is no vertical motion) ❶.

Thus:

Vertically:

$Mg + T_2 \cos\beta - T_1 \cos\alpha = 0$ ❶

$T_1 \cos\alpha - T_2 \cos\beta = Mg$ as reqd. (eqn ①)

Horizontally:

$T_1 \sin\alpha + T_2 \sin\beta = Mr\omega^2$ as reqd. (eqn ②)

As all the information is give, the marks here are for explaining the resolution of forces!

Many just wrote the equations already given in the question – this gained no marks!

iii. To find the expression for T_2:

①$\times(\sin\alpha)$: $T_1 \sin\alpha\cos\alpha - T_2 \sin\alpha\cos\beta = Mg\sin\alpha$ ③

②$\times(\cos\alpha)$: $T_1 \sin\alpha\cos\alpha + T_2 \sin\beta\cos\alpha = Mr\omega^2\cos\alpha$ ④

④ - ③:

$$T_2\left(\sin\beta\cos\alpha + \sin\alpha\cos\beta\right) = Mr\omega^2\cos\alpha - Mg\sin\alpha$$ ❶

$$T_2\sin(\alpha+\beta) = M\left(r\omega^2\cos\alpha - g\sin\alpha\right)$$

$$T_2 = \frac{M\left(r\omega^2\cos\alpha - g\sin\alpha\right)}{\sin(\alpha+\beta)}$$

Many were not clear as to how they were combining the equations.

To find the expression for T_1:

①$\times(\sin\beta)$: $T_1 \sin\beta\cos\alpha - T_2 \sin\beta\cos\beta = Mg\sin\beta$ ⑤

②$\times(\cos\beta)$: $T_1 \sin\alpha\cos\beta + T_2 \sin\beta\cos\beta = Mr\omega^2\cos\beta$ ⑥

⑥ + ⑤:

$$T_1\left(\sin\beta\cos\alpha + \sin\alpha\cos\beta\right) = Mr\omega^2\cos\beta + Mg\sin\beta$$ ❶

$$T_1\sin(\alpha+\beta) = M\left(r\omega^2\cos\beta + g\sin\beta\right)$$

$$T_1 = \frac{M\left(r\omega^2\cos\beta + g\sin\beta\right)}{\sin(\alpha+\beta)}$$

Finding the other tension was often omitted!

iv. While ***AP*** is taut for all values of ω, ***BP*** is taut only if $T_2 > 0$.

Thus $T_2 > 0 \Rightarrow \dfrac{M\left(r\omega^2\cos\alpha - g\sin\alpha\right)}{\sin\left(\alpha+\beta\right)} > 0$, or

$$M\left(r\omega^2\cos\alpha - g\sin\alpha\right) > 0$$

$$r\omega^2\cos\alpha - g\sin\alpha > 0$$

$$r\omega^2\cos\alpha > g\sin\alpha$$

$$\omega^2 > \frac{g\sin\alpha}{r\cos\alpha}$$

$$\omega > \sqrt{\frac{g\tan\alpha}{r}}$$ ➊

If ω falls below this value, ***BP*** goes slack and the particle will perform a circular motion in a horizontal circle of smaller radius at a great distance below ***A*** than currently, and the angle θ at ***A*** will be such that $\theta < \alpha$. ➊

This was often not interpreted correctly.

Some stopped at ω^2 - the question clearly asks for the value of ω!

The description was usually poor, often stating that the motion was no longer circular!